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# *Development of the A-process in Quaternions, with a Geometrical Application.*

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## I.—*Formulæ.*

**1.** Let us write (compare Hamilton, "Elements of Quaternions," p. 492, German edition, p. 741),

$$A\,.\,pq = \tfrac{1}{2}\,(pq - qp), \tag{1}$$

$$V.A\,.pqr = -\,V(pAqr + qArp + rApq), \tag{2}$$

$$S.A\,.pqr = S.pAqr. \tag{3}$$

We have at once from these equations

$$A\,.pq = A\,.KpKq = V.\,VpVq = -\,A\,.qp, \tag{4}$$

$$A\,.pp = 0. \tag{5}$$

$$\begin{aligned} S.pAqr = S.\,VpVqVr = S.qArp = SrApq &= -\,SqApr \\ = -\,SpArq &= -\,SrApq, \end{aligned} \tag{6}$$

$$S.pApq = 0. \tag{7}$$

$$A\,.pqr = S.pAqr - A\,.qrSp - ArpSq - ApqSr, \tag{8}$$

$$= S.pAqr - V.pAqr - V.qArp - V.rApq. \tag{9}$$

$$A\,.pqr = -A\,.qpr = A\,.qrp = -A\,.rqp = A\,.rpq = -\,A\,.prq, \tag{10}$$

$$A\,.ppq = 0. \tag{11}$$

$$S\,.pAqrs = Sp\,.SqArs - Sq\,.SrAsp + Sr\,.SsApq - Ss\,.SpAqr, \tag{12}$$

$$= Sp\,.Aqrs - Sq\,.Arsp + Sr\,.Aspq - Ss\,.Apqr, \tag{13}$$

$$= pS.qArs - qS.rAsp + rS.sApq - sSpAqr, \tag{14}$$

$$= pAqrs - qArsp + rAspq - sApqr. \tag{15}$$

$$\left.\begin{array}{rlll} & S.pAqrs = -S.qArsp = & S.rAspq = -S.sApqr, \\ = & -S.pArqs = +S.rAqsp = & -S.qAspr = +S.sAprq, \\ = & S.pArsq = -S.rAsqp = & +S.sAqpr = -S.qAprs, \\ = & -S.pAsrq = +S.sArqp = & -S.rAqps = +S.qApsr, \\ = & S.pAsqr = -S.sAqrp = & +S.qArps = -S.rApsq, \\ = & -S.pAqsr = +S.qAsrp = & -S.sArpq = +S.rApqs. \end{array}\right\} \tag{16}$$

2. These functions are visibly alternating, and are in nature determinantal. This aspect is more easily seen if we write

$$\left.\begin{array}{l} p = w_1 + x_1 i + y_1 j + z_1 k, \\ q = w_2 + x_2 i + y_2 j + z_2 k, \\ r = w_3 + x_3 i + y_3 j + z_3 k, \\ s = w_4 + x_4 i + y_4 j + z_4 k, \end{array}\right\} \tag{17}$$

whence

$$A.rs = i\,|y_3z_4| + j\,|z_3x_4| + k\,|x_3y_4|, \tag{18}$$

$$S.qArs = -\,|x_2y_3z_4|, \tag{19}$$

$$V.Aqrs = -\,i\,|w_2y_3z_4| - j\,|w_2z_3x_4| - k\,|w_2x_3y_4|, \tag{20}$$

$$S.Aqrs = -\,|x_2y_3z_4|, \tag{21}$$

$$S.pAqrs = -\,|w_1x_2y_3z_4|. \tag{22}$$

3. Let
$$t = h_1 p + h_2 q + h_3 r + h_4 s, \tag{23}$$

wherein the $h$'s are scalars. Then we deduce easily

$$\left.\begin{array}{l} S.tA.qrs = +\,h_1 S.pAqrs, \\ S.tA.rsp = -\,h_2 S.pAqrs, \\ S.tA.spq = +\,h_3 S.pAqrs, \\ S.tA.pqr = -\,h_4 S.pAqrs, \end{array}\right\} \tag{24}$$

therefore

$$t\,S.pAqrs = p\,S.tAqrs - q\,S.tArsp + r\,S.tAspq - s\,S.tApqr, \tag{25}$$

at least if $S.pAqrs \neq 0$. It is evident that this is the complete expression of any given quaternion $t$, and is a general relation holding between any five quaternions.

So also we have

$$t\,SpAqrs = -\,Sst.Apqr + Spt.Aqrs - Sqt.Arsp + Srt.Aspq. \tag{26}$$

4. We have from the definitions, if $a'$, $b'$, $x'$, etc., denote (as they will throughout the paper) scalars,

$$A(a'p+b'q)r = a'Apr + b'Aqr, \tag{27}$$

$$S.(a'p+b'q)Ars = a'SpArs + b'SqArs, \tag{28}$$

$$A.(a'p+b'q)rs = a'A.prs + b'A.qrs, \tag{29}$$

$$S.(a'p+b'q)Arst = a'S.pArst + b'S.qArst. \tag{30}$$

It is an easy corollary that if in any one of these functions, one of the quaternions is a linear scalar function of the others, the expression vanishes.

5. Conversely, if

$$A.pq = 0, \quad p = x'q + y', \tag{31}$$

$$S.pAqr = 0, \quad p = x'q + y'r + z', \tag{32}$$

$$A.pqr = 0, \quad p = x'q + y'r, \tag{33}$$

$$S.pAqrs = 0, \quad p = x'q + y'r + z's. \tag{34}$$

6. The following special cases deserve attention:

$$\left.\begin{gathered} A.ij = k, \\ A.x'j = 0, \\ S.iAjk = -1, \quad S.x'Ajk = 0, \\ A.ijk = -1 = \omega, \quad A.jk\omega = i, \quad A.k\omega i = -j, \quad A.\omega ij = k, \\ S.\omega Aijk = 1. \end{gathered}\right\} \tag{35}$$

$$A.Vp\,Vq = A.pq, \quad A.Sp\,Vq = 0, \tag{36}$$

$$S.Vp\,Vq\,Vr = SpAqr, \tag{37}$$

$$A.Vp\,Vq\,Vr = SpAqr, \tag{38}$$

$$A.Sp\,Vq\,Vr = -Sp.Aqr, \tag{39}$$

$$S.VpA.Vq\,Vr\,Vs = 0, \tag{40}$$

$$S.SpA.Vq\,Vr\,Vs = Sp.S.qArs. \tag{41}$$

7. Since

$$A.pqArst$$

is a quaternion, let us develop it in terms of $p$, $r$, $s$, $t$. Assume

$$A.pqArst = h_0p + h_1r + h_2s + h_3t.$$

Operating by $S.Arst$, we obtain at once

$$0 = h_0\,SpArst, \tag{42}$$

$$\therefore\ h_0 = 0. \tag{43}$$

Operating by $S.p$ and $S.q$, we obtain

$$\left.\begin{array}{l} 0 = h_1 Spr + h_2 Sps + h_3 Spt, \\ 0 = h_1 Sqr + h_2 Sqs + h_3 Sqt, \end{array}\right\} \qquad (44)$$

$$\therefore\ h_1 : h_2 : h_3 = (Sps\, Sqt - Sqs\, Spt) : (Spt\, Sqr - Spr\, Sqt) : (Spr\, Sqs - Sps\, Sqr). \qquad (45)$$

Hence if $h'$ is a yet undetermined scalar,

$$h'A.pqArst = \begin{vmatrix} r & s & t \\ Spr & Sps & Spt \\ Sqr & Sqs & Sqt \end{vmatrix}. \qquad (46)$$

To determine $h'$, take scalars of each side, whence

$$h'S.Apq\,Arst = \begin{vmatrix} Sr & Ss & St \\ Spr & Sps & Spt \\ Sqr & Sqs & Sqt \end{vmatrix} \qquad (47)$$

$$= \begin{vmatrix} Sr & Ss & St \\ S.Vp\,Vr & S.Vp\,Vs & S.Vp\,Vt \\ S.Vq\,Vr & S.Vq\,Vs & S.Vq\,Vt \end{vmatrix} \qquad (48)$$

$$= - Sr.S.Vp\,Vq\,V.Vs\,Vt - Ss.S.Vp\,Vq\,V.Vt\,Vr - St.S.Vp\,Vq\,V.Vr\,Vs \qquad (49)$$

$$= S.Vp\,Vq\,VArst = S.A.pqArst, \qquad (50)$$

$$\therefore\ h' = 1$$

and

$$A.pqArst = \begin{vmatrix} r & s & t \\ S.rp & S.sp & S.tp \\ S.rq & S.sq & S.tq \end{vmatrix}. \qquad (51)$$

This formula is strictly analogous to the vector formula

$$V.\alpha V\beta\gamma = - \begin{vmatrix} \beta & \gamma \\ S\beta\alpha & S\gamma\alpha \end{vmatrix}.$$

This result might have been obtained likewise by direct expansion:

$$\begin{aligned} A.pqA.rst &= S.ApqA.rst + V.ApqA\,.rst \\ &= - S.Vp\,Vq\,(Sr.V.Vs\,Vt + Ss.V.Vt\,Vr + St.V.Vr\,Vs) \\ &\quad - S.Vr\,Vs\,Vt.V.Vp\,Vq \\ &\quad + Sp.V.Vq\,(Sr.V.Vs\,Vt + Ss.V.Vt\,Vr + St.V.Vr\,Vs) \\ &\quad - Sq.V.Vp\,(Sr.V.Vs\,Vt + Ss.V.Vt\,Vr + St.V.Vr\,Vs). \end{aligned}$$

Now the second line is equal to

$$-Vr\,.\,S.V.Vs\,Vt.V.Vp\,Vq - Vs\,.\,S.V.Vt\,Vr\,V.Vp\,Vq - Vt\,.\,S.V.Vr\,Vs\,V.Vp\,Vq\,.$$

The last two combine into

$$\begin{aligned} Vr\,.\,(&+Sp\,.\,Ss\,.\,S\,.\,Vq\,Vt - Sp\,.\,St\,.\,S\,.\,Vq\,Vs \\ &- Sq\,.\,Ss\,.\,S\,.\,Vp\,Vt + Sq\,.\,St\,.\,S\,.\,Vp\,Vs) \\ &+ \text{etc.,} \end{aligned}$$

which, added to the reduced expression preceding, is

$$= \begin{vmatrix} Vr & Vs & Vt \\ S.pr & S.ps & S.pt \\ S.qr & S.qs & S.qt \end{vmatrix}.$$

The first line is easily seen to be the determinant

$$\begin{vmatrix} Sr & Ss & St \\ S.pr & S.ps & S.pt \\ S.qr & S.qs & S.qt \end{vmatrix}.$$

Whence

$$A\,.pqA\,.\,rst = \begin{vmatrix} r & s & t \\ Spr & Sps & Spt \\ Sqr & Sqs & Sqt \end{vmatrix}.$$

If we expand each side of the equality $A\,.\,pA\,.\,qrsA.stu = -A\,.pA\,.\,stuA.qrs$ by this theorem, we get

$$\begin{aligned} &Sps\,.\,(qS\,.\,sA\,.\,tur - rS\,.\,sA\,.\,tuq + tS.\,sA\,.\,qru - uS.\,sA\,.\,qrt) \\ &= S.\,(Spq\,.\,S.sA.tur - Spr\,.\,S.sA.tuq + S.pt.S.\,sA\,.\,qru - S.pu.S.sA\,.\,qrt). \end{aligned}$$

Since $p$ may be any quaternion, let it be $A\,.qrt$, whence

$$s\,.\,S.\,uA\,.\,qrt = uS\,.\,sA\,.\,qrt - qS\,.\,sA\,.\,rtu + rS\,.\,sA\,.\,tuq - tS.\,sA\,.\,uqr,$$

which is the deduction of this important formula analogous to Hamilton's deduction of the corresponding vector formula. From this formula, following Hamilton's process for vectors, we may deduce several others, showing relations between quaternions. Thus

$$A\,.\,pAqrs\,Auvw = \begin{vmatrix} u & v & w \\ SuAqrs & SvAqrs & SwAqrs \\ Spu & Spv & Spw \end{vmatrix} = -\begin{vmatrix} q & r & s \\ SqAuvw & SrAuvw & SsAuvw \\ Sqp & Srp & Ssp \end{vmatrix},$$

a relation connecting any six quaternions.

8. We have from (51)

$$A\,.\,Auvw\,Axyz\,Arst$$

$$= \begin{vmatrix} r & s & t \\ S.rAuvw & S.sAuvw & S.tAuvw \\ S.rAxyz & S.sAxyz & S.tAxyz \end{vmatrix} = \begin{vmatrix} u & v & w \\ S.uAxyz & S.vAxyz & S.wAxyz \\ S.uArst & S.vArst & S.wArst \end{vmatrix}$$

$$= \begin{vmatrix} x & y & z \\ S.xArst & S.yArst & S.zArst \\ S.xAuvw & S.yAuvw & S.zAuvw \end{vmatrix} \tag{52}$$

Operating on (51) by $S.u$

$$S.uApqArst = \begin{vmatrix} S.ru & S.su & S.tu \\ S.rp & S.sp & S.tp \\ S.rq & S.sq & S.tq \end{vmatrix} \tag{53}$$

$$= S.rA\,.\,stAupq \tag{54}$$

$$= -\,S(A\,.\,rst\,.\,Aupq), \tag{55}$$

$$\therefore \quad -\,S^2.A\,.\,rst = \begin{vmatrix} S.r^2 & S.rs & S.rt \\ S.rs & S.s^2 & S.st \\ S.rt & S.st & S.t^2 \end{vmatrix}. \tag{56}$$

Operating on (52) by $S.A\,.\,lmn$, we have

$$S.Almn\,A\,.\,Auvw\,Axyz\,Arst = \begin{vmatrix} S.rAlmn & S.sAlmn & S.tAlmn \\ S.rAuvw & S.sAuvw & S.tAuvw \\ S.rAxyz & S.sAxyz & S.tAxyz \end{vmatrix}. \tag{57}$$

From (58), making

$$u = m,\quad v = n,\quad w = p,$$
$$x = n,\quad y = p,\quad z = l,$$
$$r = p,\quad s = l,\quad t = m,$$

we have

$$S.Almn\,A\,.\,Amnp\,A\,.\,npl\,A\,.\,plm = (S.lAmnp)^3, \tag{58}$$

and from (52) in like manner

$$A\,.\,Almn\;Amnp\;Anpl = -\,n\,S^2.\,lAmnp. \tag{59}$$

These two are strictly analogous to

$$\begin{aligned} V.\,V\alpha\beta\;V\beta\gamma &= -\,\beta S\alpha\beta\gamma, \\ S.\,V\alpha\beta\;V\beta\gamma\;V\gamma\alpha &= -\,S^2.\,\alpha\beta\gamma. \end{aligned}$$

9. We deduce easily

$$S.\,(l+m)\,A\,(m+n)(n+p)(p+l) = 0, \tag{60}$$

$$S.\,(l+m+n)\,A\,.\,(m+n+p)(n+p+l)(p+l+m) = 3S\,.\,lAmnp, \tag{61}$$

$$\begin{aligned} S\,.\,A\,(l+m+n)(m+n+p)(n+p+l)\;A\,.\,A\,&(m+n+p)(n+p+l)(p+l+m) \\ A\,&(n+p+l)(p+l+m)(l+m+n) \\ A\,&(p+l+m)(l+m+n)(m+n+p) \\ &= 27S^3.\,lAmnp, \end{aligned} \tag{62}$$

$$\begin{aligned} S\,.\,A\,(l+m)(m+n)(n+p)\;A\,.\,A\,&(m+n)(n+p)(p+l) \\ A\,&(n+p)(p+l)(l+m) \\ A\,&(p+l)\,(l+m)\,(m+n) = 0, \end{aligned} \tag{63}$$

$$\begin{aligned} S\,.\,A\,.\,(A\,.\,lmn\,.\,A\,.\,mnp\,.\,A\,.\,npl)\;A\,.\,A\,.\,&(A\,.\,mnp\,.\,A\,.\,npl\,.\,A\,.\,plm) \\ A\,.\,&(A\,.\,npl\,.\,A\,.\,plm\,.\,A\,.\,lmn) \\ A\,.\,&(A\,.\,plm\,.\,A\,.\,lmn\,.\,A\,.\,mnp) = S^9.\,lAmnp. \end{aligned} \tag{64}$$

Formulas like these, of course, express determinant theorems, and might be written out in very elaborate form in that way. It is evident they can be developed indefinitely. It may be as well, however, to remark here, that as their vector analogs express theorems in spherical trigonometry, so these can be used as proofs of theorems in space of four dimensions analogous to those in the geometry of the sphere.

For any ten quaternions whatever

$$\begin{vmatrix} Spp_1 & Spq_1 & Spr_1 & Sps_1 & Spt_1 \\ Sqp_1 & Sqq_1 & Sqr_1 & Sqs_1 & Sqt_1 \\ Srp_1 & Srq_1 & Srr_1 & Srs_1 & Srt_1 \\ Ssp_1 & Ssq_1 & Ssr_1 & Sss_1 & Sst_1 \\ Stp_1 & Stq_1 & Str_1 & Sts_1 & Stt_1 \end{vmatrix} = 0. \tag{65}$$

For, if we multiply the first row by $S\,.\,qArst$, the second by $S\,.\,rAstp$, the third by $S.sAtpq$, the fourth by $S.tApqr$, the fifth by $S.pAqrs$, and then add the last four rows to the first, the first row will vanish.

Again, if for $Spp_1$ we write $Sp_0p_1$ the determinant reduces to

$$S.p_1(p_0-p)\begin{vmatrix} Sqq_1 & Sqr_1 & Sqs_1 & Sqt_1 \\ Srq_1 & Srr_1 & Srs_1 & Srt_1 \\ Ssq_1 & Ssr_1 & Sss_1 & Sst_1 \\ Stq_1 & Str_1 & Sts_1 & Stt_1 \end{vmatrix}, \tag{66}$$

which being resolved, and (53) applied, becomes

$$\begin{aligned} &= Sp_1(p_0-p)(Sqq_1\,S.rAstAr_1s_1t_1 - Srq_1\,S.stAq\,Ar_1s_1t_1 \\ &\qquad + S.sq_1\,S.tAqr\,Ar_1s_1t_1 - Stq_1\,.\,S.tq\,Ars\,Ar_1s_1t_1) \qquad (67) \\ &= -S.p_1(p_0-p)\,S.q_1Ar_1s_1t_1\,S.qArst. \qquad (68) \end{aligned}$$

We have incidentally proved here that

$$\begin{vmatrix} Sqq_1 & Sqr_1 & Sqs_1 & Sqt_1 \\ Srq_1 & Srr_1 & Srs_1 & Srt_1 \\ Ssq_1 & Ssr_1 & Sss_1 & Sst_1 \\ Stq_1 & Str_1 & Sts_1 & Stt_1 \end{vmatrix} = -S.qArst\,S.q_1Ar_1s_1t_1, \tag{69}$$

and as a corollary

$$S^2.qArst = -\begin{vmatrix} S.q^2 & Sqr & Sqs & Sqt \\ S.qr & S.r^2 & Srs & Srt \\ S.qs & Srs & S.s^2 & Sst \\ S.qt & Srt & Sst & S.t^2 \end{vmatrix} \tag{70}$$

10. Let us consider the set of quaternions

$$\left.\begin{aligned} l_1 &= \tfrac{1}{2}\theta^{-\frac{1}{2}}(\theta+i+j+k), \\ l_2 &= \tfrac{1}{2}\theta^{-\frac{1}{2}}(\theta-i-j+k), \\ l_3 &= \tfrac{1}{2}\theta^{-\frac{1}{2}}(\theta-i+j-k), \\ l_4 &= \tfrac{1}{2}\theta^{-\frac{1}{2}}(\theta+i-j-k), \\ \theta &= \sqrt{-1}. \end{aligned}\right\} \tag{71}$$

We have from them at once the equations

$$\left.\begin{aligned} S.l_r^2 &= \theta, \\ S.l_rl_s &= 0. \end{aligned}\right\} \tag{72}$$

$$A.l_1l_2l_3 = -l_4, \quad A.l_2l_3l_4 = l_1, \quad A.l_3l_4l_1 = -l_2, \quad A.l_4l_1l_2 = l_3. \tag{73}$$

$$S.l_1Al_2l_3l_4 = \theta. \tag{74}$$

The latter equation shows that they are linearly independent. The symmetry of the equations is apparent. These four bear a close analogy to the vector $i, j, k$.

11. The following formulæ are of use:

$$\left.\begin{aligned} \theta &= \tfrac{1}{2}\theta^{\frac{1}{2}}(l_1 + l_2 + l_3 + l_4), \\ i &= \tfrac{1}{2}\theta^{\frac{1}{2}}(l_1 - l_2 - l_3 + l_4), \\ j &= \tfrac{1}{2}\theta^{\frac{1}{2}}(l_1 - l_2 + l_3 - l_4), \\ k &= \tfrac{1}{2}\theta^{\frac{1}{2}}(l_1 + l_2 - l_3 - l_4), \end{aligned}\right\} \tag{75}$$

$$\left.\begin{aligned} \theta^{\frac{1}{2}} &= \tfrac{1}{2}(l_1 + l_2 + l_3 + l_4), \\ \theta &= -1 + \tfrac{1}{2}\sqrt{2}(l_1 + l_2 + l_3 + l_4). \end{aligned}\right\} \tag{76}$$

$$\left.\begin{aligned} 4\theta &= l_1^2 + l_2^2 + l_3^2 + l_4^2, \\ i &= l_1^2 - l_2^2 - l_3^2 + l_4^2, \\ j &= l_1^2 - l_2^2 + l_3^2 - l_4^2, \\ k &= l_1^2 + l_2^2 - l_3^2 - l_4^2. \end{aligned}\right\} \tag{77}$$

$$\left.\begin{aligned} l_1^2 &= \tfrac{1}{8}\theta^{\frac{1}{2}}(7l_1 + 3l_2 + 3l_3 + 3l_4), \\ l_2^2 &= \tfrac{1}{8}\theta^{\frac{1}{2}}(3l_1 + 7l_2 + 3l_3 + 3l_4), \\ l_3^2 &= \tfrac{1}{8}\theta^{\frac{1}{2}}(3l_1 + 3l_2 + 7l_3 + 3l_4), \\ l_4^2 &= \tfrac{1}{8}\theta^{\frac{1}{2}}(3l_1 + 3l_2 + 3l_3 + 7l_4). \end{aligned}\right\} \tag{78}$$

$$\left.\begin{aligned} A.l_1l_2 &= \tfrac{1}{2}\theta(-i + j) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_3 - l_4), \\ A.l_1l_3 &= \tfrac{1}{2}\theta(\phantom{-}i - k) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_4 - l_2), \\ A.l_4l_1 &= \tfrac{1}{2}\theta(\phantom{-}j - k) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_3 - l_2), \\ A.l_2l_3 &= \tfrac{1}{2}\theta(\phantom{-}j + k) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_1 - l_4), \\ A.l_2l_4 &= \tfrac{1}{2}\theta(-i - k) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_3 - l_1), \\ A.l_3l_4 &= \tfrac{1}{2}\theta(\phantom{-}i + j) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_1 - l_2). \end{aligned}\right\} \tag{79}$$

$$\left.\begin{aligned} V.l_1 &= \tfrac{1}{4}(\phantom{-}3l_1 - l_2 - l_3 - l_4), \\ V.l_2 &= \tfrac{1}{4}(-l_1 + 3l_2 - l_3 - l_4), \\ V.l_3 &= \tfrac{1}{4}(-l_1 - l_2 + 3l_3 - l_4), \\ V.l_4 &= \tfrac{1}{4}(-l_1 - l_2 - l_3 + 3l_4). \end{aligned}\right\} \tag{80}$$

12. Since $l_1$, $l_2$, $l_3$, $l_4$ are linearly independent, any other quaternion $q$ may be expressed in terms of them. Suppose then

$$q = h'l_1 + h''l_2 + h'''l_3 + h^{\text{IV}}l_4. \tag{81}$$

We have at once

$$h^{(r)} = -\theta . S . l_r q, \tag{82}$$

$$S . q = \tfrac{1}{2}\theta^{\frac{1}{2}}(h' + h'' + h''' + h^{\text{IV}}), \tag{83}$$

$$S . q^2 = \theta(h'^2 + h''^2 + h'''^2 + h^{\text{IV}2}), \tag{84}$$

$$\begin{aligned} Vq = \tfrac{1}{4}l_1(3h' - h'' - h''' - h^{\text{IV}}) + \tfrac{1}{4}l_2(-h' + 3h'' - h''' - h^{\text{IV}}) \\ + \tfrac{1}{4}l_3(-h' - h'' + 3h''' - h^{\text{IV}}) + \tfrac{1}{4}l_4(-h' - h'' - h''' + 3h^{\text{IV}}). \end{aligned} \tag{85}$$

13. If

$$r = \Sigma . g'l_1, \tag{85}$$

$$S . qr = \theta\Sigma . h'g', \tag{86}$$

$$A . qr = A . l_1 l_2 |h'g''| + \text{etc.} \tag{87}$$

So also

$$A . qrs = l_1|h''g'''f^{\text{IV}}| - l_2|h'''g^{\text{IV}}f'| + l_3|h^{\text{IV}}g'f''| - l_4|h'g''f'''|. \tag{88}$$

$$S . qArst = \theta . |h'g''f'''e^{\text{IV}}|. \tag{89}$$

14. We solve the linear quaternion operator by these functions thus. Let

$$r = \Phi . q. \tag{90}$$

Let $s$, $t$, $u$ be any quaternions satisfying the equation

$$\Phi q = A . stu, \tag{91}$$

$$\therefore\ S . s\Phi q = 0,\ \ S . t\Phi q = 0,\ \ S . u\Phi q = 0. \tag{92}$$

Now these three may also be written

$$S . q\Phi's = 0,\ \ S . q\Phi't = 0,\ \ S . q\Phi'u = 0, \tag{93}$$

$$\therefore\ mq = A . \Phi's\,\Phi't\,\Phi'u. \tag{94}$$

Operate by $S . \Phi'v$, $\therefore$

$$mS . q\Phi'v = S . \Phi'v\,A . \Phi's\,\Phi't\,\Phi'u. \tag{95}$$

Whence

$$m = \frac{S.\Phi' s A \Phi' t \Phi' u \Phi' v}{S.sAtuv}. \tag{96}$$

In this equation it is evident that $s$, $t$, $u$, $v$ may be *any* linearly independent quaternions. For if for either we substitute an arbitrary linear function of all four, we find the expression reducing to exactly the same again as (96).

Put $\Phi - g$ for $\Phi$, whence

$$m_g q = m_g(\Phi - g)^{-1} A.stu \tag{97}$$

$$= A(\Phi' - g)s(\Phi' - g)t(\Phi' - g)u, \tag{98}$$

$$\begin{aligned} &= A.\Phi' s\Phi' t\Phi' u - g(A.s\Phi' t\Phi' u + A.t\Phi' u\Phi' s + A.u\Phi' s\Phi' t) \\ &+ g^2(A.st\Phi' u + A.tu\Phi' s + A.us\Phi' t) - g^3 A.stu. \end{aligned} \tag{99}$$

Here

$$m_g = m - m_1 g + m_2 g^2 - m_3 g^3 + g^4, \tag{100}$$

where

$$m_1 = \frac{S.sA\Phi' t\Phi' u\Phi' v + S.\Phi' sA.t\Phi' u\Phi' v + S.\Phi' sA\Phi' tu\Phi' v + S.\Phi' sA\Phi' t\Phi' uv}{S.sAtuv}, \tag{101}$$

$$m_2 = \frac{\begin{aligned} &S.sAt\Phi' u\Phi' v + S.sA\Phi' tu\Phi' v + S.sA.\Phi' t\Phi' uv \\ &\qquad + S.\Phi' sA.tu\Phi' v + S.\Phi' sA.t\Phi' uv + S.\Phi' sA\Phi' tuv \end{aligned}}{S.sAtuv}, \tag{102}$$

$$m_3 = \frac{S.sAtu\Phi' v + S.sAt\Phi' uv + S.sA\Phi' tuv + S.\Phi' sAtuv}{S.sAtuv}. \tag{103}$$

Substitute these values, operate by $\Phi - g$ and equate like powers of $g$:

$$m\,A.stu = \Phi A.\Phi' s\Phi' t\Phi' u, \tag{104}$$

$$m_1 A.stu = \Phi(A.s\Phi' t\Phi' u + A.t\Phi' u\Phi' s + A.u\Phi' s\Phi' t) + A.\Phi' s\Phi' t\Phi' u, \tag{105}$$

$$\begin{aligned} m_2 A.stu = {} & \Phi(A.st\Phi' u + A.tu\Phi' s + A.us\Phi' t) \\ & + (A.s\Phi' t\Phi' u + A.t\Phi' u\Phi' s + A.u\Phi' s\Phi' t), \end{aligned} \tag{106}$$

$$m_3 A.stu = \Phi(A.stu) + (A.st\Phi' u + A.tu\Phi' s + A.us\Phi' t). \tag{107}$$

It is plain from (104) and (105) that the parenthesis in (105) is a function of $A.stu$, say

$$\mathrm{X}.Astu = (A.s\Phi' t\Phi' u + A.t\Phi' u\Phi' s + A.u\Phi' s\Phi' t). \tag{108}$$

Likewise we may write

$$\Psi A.stu = A.st\Phi' u + A.tu\Phi' s + A.us\Phi' t; \tag{109}$$

$$\therefore\ m_1 = \Phi\mathrm{X} + m\Phi^{-1}, \tag{110}$$

$$m_2 = \Phi\Psi + \mathrm{X}, \tag{111}$$

$$m_3 = \Phi + \Psi, \tag{112}$$

whence

$$\Psi = m_3 - \Phi, \tag{113}$$

$$\mathrm{X} = m_2 - m_3\Phi + \Phi^2, \tag{114}$$

$$m\Phi^{-1} = m_1 - m_2\Phi + m_3\Phi^2 - \Phi^3, \tag{115}$$

$$\Phi^4 - m_3\Phi^3 + m_2\Phi^2 - m_1\Phi + m = 0. \tag{116}$$

15. If $m_3$ vanish, (107) gives

$$\Phi . Astu = -(A . st\Phi' u + A . tu\Phi' s + A . us\Phi' t). \tag{117}$$

If $m_2$ vanish, (106) gives

$$\Phi(A . st\Phi' u + A . tu\Phi' s + A . us\Phi' t) = -(A . s\Phi' t\Phi' u + A . t\Phi' u\Phi' s + A . u\Phi' s\Phi' t). \tag{118}$$

If $m_1$ vanish,

$$A . \Phi' s\Phi' t\Phi' u = \Phi(A . s\Phi' t\Phi' u + A . t\Phi' u\Phi' s + A . u\Phi' s\Phi' t). \tag{119}$$

If $m$ vanish,

$$\Phi A . \Phi' s\Phi' t\Phi' u = 0. \tag{120}$$

If $m$ and $m_1$ vanish, we have in addition to (120),

$$\Phi^2(A . s\Phi' t\Phi' u + A . t\Phi' u\Phi' s + A . u\Phi' s\Phi' t) = 0. \tag{121}$$

If also $m_2$ vanishes, we may add the equation

$$\Phi^3(A . st\Phi' u + A . tu\Phi' s + A . us\Phi' t) = 0. \tag{122}$$

And finally, if $m_3$ is also zero,

$$\Phi^4 . A . stu = 0. \tag{123}$$

16. If $\Phi$ is self-transverse, it may be written

$$\Phi = -g_1\theta l_1 Sl_1 + g_2\theta l_2 Sl_2 - g_3\theta l_3 Sl_3 + g_4\theta l_4 Sl_4, \tag{124}$$

where $l_1, l_2, l_3, l_4$ are as in §10.

For let $q_1, q_2, q_3, q_4$ be the axes of $\Phi$, then

$$\Phi q_1 = g_1 q_1, \text{ etc.}, \tag{125}$$

$$\left.\begin{aligned} \therefore\ S . q_2\Phi q_1 &= g_1 Sq_1q_2, \\ S . q_1\Phi q_2 &= g_2 Sq_1q_2, \end{aligned}\right\} \tag{126}$$

$$\therefore\ (g_1 - g_2)\, S . q_1q_2 = 0, \tag{127}$$

and if $g_1$, $g_2$, $g_3$, $g_4$ are all different,

$$S.q_1q_2 = 0, \text{ etc.} \tag{128}$$

Further,

$$A.\Phi q_1\Phi q_2\Phi q_3 = m\Phi^{-1}A.q_1q_2q_3 = g_1g_2g_3Aq_1q_2q_3 = g_1g_2g_3g_4\Phi^{-1}A.q_1q_2q_3, \tag{129}$$

$$\therefore\ \Phi Aq_1q_2q_3 = g_4Aq_1q_2q_3 = g_4q_4. \tag{130}$$

Likewise

$$\Phi A.q_2q_3q_4 = g_1A.q_2q_3q_4 = -g_1(-q_1). \tag{131}$$

Hence the axes satisfy such relations as those defining the $l$'s of §10.

But we may also write $\Phi$ in the form

$$\Phi = g_1S.1 - g_2iSi - g_3jSj - g_4kSk, \tag{132}$$

since

$$S.1i = Sij = 0, \text{ etc.} \tag{133}$$

and

$$\left.\begin{aligned} \Phi.A1ij &= -g_4k, & -k &= A.1ij, \\ \Phi.Aijk &= -g_11, & -1 &= A.ijk, \\ \Phi.Ajk1 &= -g_2i, & -i &= A.jk1, \\ \Phi.A.k1i &= -g_3j, & -j &= A.k1i. \end{aligned}\right\} \tag{134}$$

17. If some of the roots are equal, results hold analogous to those in the theory of the linear vector operator by much the same proofs.

18. The general expression for $\Phi$ in any case, in terms of its roots and axes, is

$$\Phi = \frac{g_1q_1S()A.q_2q_3q_4 - g_2q_2S()Aq_3q_4q_1 + g_3q_3S()Aq_4q_1q_2 - g_3q_4S()Aq_1q_2q_3}{S.q_1Aq_2q_3q_4}, \tag{135}$$

whence $\Phi'$ may be written

$$\Phi' = \frac{\begin{aligned} &g_1A.q_2q_3q_4Sq_1() - g_2A.q_3q_4q_1S.q_2() \\ &\qquad + g_3A.q_4q_1q_2S.q_3() - g_4A.q_1q_2q_3Sq_4() \end{aligned}}{S.q_1Aq_2q_3q_4}. \tag{136}$$

19. If we write

$$\left.\begin{aligned} K_1 &= \phantom{-}\frac{q_1S()Aq_2q_3q_4}{S.q_1Aq_2q_3q_4}, \\ K_2 &= -\frac{q_2S()Aq_3q_4q_1}{S.q_1Aq_2q_3q_4}, \\ K_3 &= \phantom{-}\frac{q_3S()Aq_4q_1q_2}{S.q_1Aq_2q_3q_4}, \\ K_4 &= -\frac{q_4S()Aq_1q_2q_3}{S.q_1Aq_2q_3q_4}, \end{aligned}\right\} \tag{137}$$

whence
$$K_r^2 = K_r, \quad K_r K_s = 0; \tag{138}$$
and
$$I = K_1 + \theta K_2 - K_3 - \theta K_4, \tag{139}$$
whence
$$I^2 = K_1 - K_2 + K_3 - K_4, \tag{140}$$
$$I^3 = K_1 - \theta K_2 - K_3 + \theta K_4, \tag{141}$$
$$I^4 = K_1 + K_2 + K_3 + K_4 = 1; \tag{142}$$
we have
$$\Phi = g_1 K_1 + g_2 K_2 + g_3 K_3 + g_4 K_4, \tag{143}$$
$$= a + bI + cI^2 + dI^3. \tag{144}$$

In this equation

$$\left.\begin{aligned} a &= \tfrac{1}{4}(g_1 + g_2 + g_3 + g_4), \\ b &= \tfrac{1}{4}(g_1 - \theta g_2 - g_3 + \theta g_4), \\ c &= \tfrac{1}{4}(g_1 - g_2 + g_3 - g_4), \\ d &= \tfrac{1}{4}(g_1 + \theta g_2 - g_3 - \theta g_4). \end{aligned}\right\} \tag{145}$$

The general theory of these operators may thence be developed in the same manner as I have developed that of the linear *vector* operator (read before the American Math. Soc., Aug. 1895). The quaternion operator may also be written in a sedenion form, and many theorems thence deduced.

20. In the equation
$$m\Phi^{-1}A\,.\,stu = A\,.\,\Phi's\,\Phi't\,\Phi'u,$$

substitute $s$ for $\Phi's$, $\Phi'^{-1}s$ for $s$, etc., then change $\Phi$ to $\Phi - g$. We have
$$m_g A\,.\,(\Phi' - g)^{-1}s\,(\Phi' - g)\,t\,(\Phi' - g)\,u = (\Phi - g)\,A\,.\,stu. \tag{146}$$

Since

$$\begin{aligned} m_g(\Phi - g)^{-1}A\,.\,pqr &= A\,.\,(\Phi' - g)\,p\,(\Phi' - g)\,q\,(\Phi' - g)\,r \\ &= (m\Phi^{-1} - g\mathrm{X} + g^2\Psi - g^3)\,A\,.\,pqr, \end{aligned} \tag{147}$$

$$\left.\begin{aligned} \therefore\ \frac{m_g}{g}S\,.\,q\,(\Phi - g)^{-1}q &= \frac{m}{g}S\,.\,q\Phi^{-1}q - Sq\mathrm{X}q + gSq\Psi q - g^2S\,.\,q^2, \\ \frac{m_h}{h}S\,.\,q\,(\Phi - h)^{-1}q &= \frac{m}{h}S\,.\,q\Phi^{-1}q - Sq\mathrm{X}q + hSq\Psi q - h^2S\,.\,q^2, \end{aligned}\right\} \tag{148}$$

$$\begin{aligned} \therefore\ \frac{m_g}{g}S\,.\,q\,(\Phi - g)^{-1}q &- \frac{m_h}{h}S\,.\,q\,(\Phi - h)^{-1}q \\ &= (g - h)\left\{S\,.\,q\Psi q - (g + h)\,S\,.\,q^2 - \frac{mS\,.\,q\Phi^{-1}q}{gh}\right\}. \end{aligned} \tag{149}$$

These are identical when $g = h$, and also when

$$ghSq\Psi q - gh(g+h)\,S.q^2 - mSq\Phi^{-1}q = 0. \qquad (150)$$

Thus whatever $g$ or $h$, they are identical if

$$S.q\Psi q = 0, \quad S.q^2 = 0, \quad \text{and} \quad mSq\Phi^{-1}q = 0. \qquad (151)$$

These equations are of use in the theory of surfaces of the second order.

21. Again, if

$$\left.\begin{aligned} S.q(\Phi - g)^{-1}q &= 0, \\ S.q(\Phi - h)^{-1}q &= 0, \end{aligned}\right\} \qquad (152)$$

$$\left.\begin{aligned} \therefore\ mSq\Phi^{-1}q - gSq\mathrm{X}q + g^2Sq\Psi q - g^3S.q^2 &= 0, \\ mSq\Phi^{-1}q - hSq\mathrm{X}q + h^2Sq\Psi q - h^3S.q^2 &= 0. \end{aligned}\right\} \qquad (153)$$

Hence whatever $x$,

$$m(1-x)\,Sq\Phi^{-1}q - (g - hx)\,Sq\mathrm{X}q + (g^2 - h^2x)\,Sq\Psi q - (g^3 - h^3x)\,Sq^2 = 0. \qquad (154)$$

If $x = 1$, $\quad Sq\mathrm{X}q + (g+h)\,Sq\Psi q - (g^2 + gh + h^2)\,Sq^2 = 0. \qquad (155)$

If $x = g/h$, $\quad mSq\Phi^{-1}q - gSq\Psi q + g(g+h)\,S.q^2 = 0. \qquad (156)$

If $x = g^2/h^2$, $\quad m(g+h)\,Sq\Phi^{-1}q - ghSq\mathrm{X}q - g^2h^2S.q^2 = 0. \qquad (157)$

If $x = g^3/h^3$, $\quad m(g^2 + gh + h^2)\,Sq\Phi^{-1}q - gh(g+h)\,Sq\mathrm{X}q + g^2h^2Sq\Psi q = 0. \qquad (158)$

22. Passing now to considerations of a different kind, let

$$q = x'l_1 + x''l_2 + x'''l_3 + x^{\mathrm{IV}}l_4. \qquad (159)$$

Let

$$\Box = l_1\frac{\partial}{\partial x'} + l_2\frac{\partial}{\partial x''} + l_3\frac{\partial}{\partial x'''} + l_4\frac{\partial}{\partial x^{\mathrm{IV}}}. \qquad (160)$$

Then we have

$$\begin{aligned}
\Box.q &= 4\theta, & (161)\\
\Box.Sq &= \tfrac{1}{2}\,\theta^{\frac{1}{2}}(l_1 + l_2 + l_3 + l_4) = \theta, & (162)\\
\Box.Vq &= 3\theta, & (163)\\
\Box.T^2q &= 2\theta Sq - 6\theta Vq, & (164)\\
\Box.Saq &= \theta a. & (165)\\
\Box.aq &= 4\theta Sa, & (166)\\
\Box.Vaq &= \theta(-3Sa + Va), & (167)\\
\Box.A.aq &= -2\theta.Va, & (168)
\end{aligned}$$

etc.

## II.—*A Geometrical Application of the use of the A-Functions.*

1. Let $ABCD$ be any four non-coplanar points in space. Let us denote any point in space, as $P$, by the quaternion

$$p = \frac{x' l_1 + x'' l_2 + x''' l_3 + x^{\text{IV}} l_4}{\Sigma . x'}, \tag{1}$$

wherein

$$x' : x'' : x''' : x^{\text{IV}} = (P\text{–}BCD) : (P\text{–}CDA) : (P\text{–}DAB) : (P\text{–}ABC). \tag{2}$$

2. Evidently $A$, $B$, $C$, $D$ are denoted respectively by

$$l_1, l_2, l_3, l_4.$$

3. The areas of the faces opposite $A$, $B$, $C$, $D$, being denoted by $a'$, $b'$, $c'$, $d'$, the perpendiculars from $P$ on the faces are evidently as

$$\frac{x'}{a'} : \frac{x''}{b'} : \frac{x'''}{c'} : \frac{x^{\text{IV}}}{d'}.$$

4. If $Q$ be any second point $(y', y'', y''', y^{\text{IV}})$, any point $R$ on the line joining $P$ and $Q$ is

$$r = \frac{m'p + n'q}{m' + n'}, \text{ where } \overline{PR} : \overline{RQ} = n' : m'.$$

The proof is easy by considering the relations of the perpendiculars involved.

This equation is exactly equivalent to

$$A . pqr = 0, \tag{3}$$

which is therefore the equation of the right line through $p$ and $q$.

5. The centroid $G$ of the fundamental tetrahedron is

$$\tfrac{1}{4}(l_1 + l_2 + l_3 + l_4) = g. \tag{4}$$

6. If $q$ be any point,

$$q = Sq + Vq. \tag{5}$$

But

$$Sq = (x' + x'' + x''' + x^{\text{IV}})\, g, \tag{6}$$

hence

$$\Sigma x' = \frac{S . q}{g}. \tag{7}$$

When $S.q = 0$,

$$\Sigma x' = 0 \tag{8}$$

and the point $Vq$ is at infinity. Hence $Vq$ is the point at infinity on the line joining $G$ and $Q$.

7.
$$Kq = 2Sq - q = 2g - q \tag{9}$$

is evidently such a point that $G$ bisects the segment joining it to $Q$.

8. If three points $a_1$, $a_2$, $a_3$ be given, any point in the plane joining them is

$$p = x'a_1 + y'a_2 + z'a_3. \tag{10}$$

This is exactly equivalent to

$$S.pAa_1a_2a_3 = 0, \tag{11}$$

which is therefore the equation of the plane through $a_1$, $a_2$, $a_3$.

9. It is easy to see that, if lines be drawn from $A$, $B$, $C$, $D$ through a point $P$, to cut the opposite faces in $P_A$, $P_B$, $P_C$, $P_D$, and then lines be drawn through each of these points and the three vertices in the same plane with it to intersect the edges in $P_{AB}$, $P_{AC}$, $P_{AD}$, etc., then

$$\overline{BP_{AD}} : \overline{P_{AD}C} = x''' : x'', \text{ etc.,}$$

and $P_{AD} = x''l_2 + x'''l_3$.

10. If we define as the isotomic conjugate of $P(p = x'l_1 + x''l_2 + x'''l_3 + x^{\text{IV}}l_4)$ the point ${}_iP$,

$${}_iP = \frac{1}{x'}\,l_1 + \frac{1}{x''}\,l_2 + \frac{1}{x'''}\,l_3 + \frac{1}{x^{\text{IV}}}\,l_4, \tag{12}$$

then the harmonic conjugates of the points ${}_iP_{AB}$, ${}_iP_{AC}$, ${}_iP_{AD}$, ${}_iP_{BC}$, ${}_iP_{BD}$, ${}_iP_{CD}$ are represented respectively by $\frac{1}{x'''}\,l_3 - \frac{1}{x^{\text{IV}}}\,l_4$, $\frac{1}{x''}\,l_2 - \frac{1}{x^{\text{IV}}}\,l_4$,

$$\frac{1}{x''}\,l_2 - \frac{1}{x'''}\,l_3,\ \frac{1}{x'}\,l_1 - \frac{1}{x^{\text{IV}}}\,l_4,\ \frac{1}{x'}\,l_1 - \frac{1}{x'''}\,l_3,\ \frac{1}{x'}\,l_1 - \frac{1}{x''}\,l_2.$$

It is plain at a glance that any four of these lie in one plane, that is, all of them lie in a plane, namely, the plane

$$S.r\,(x'l_1 + x''l_2 + x'''l_3 + x^{\text{IV}}l_4) = 0. \tag{13}$$

This plane is the polar plane of the point ${}_iP$ with respect to the tetrahedron $ABCD$, and we may call it the *isopolar* plane of $P$,

$$x'l_1 + x''l_2 + x'''l_3 + x^{\text{IV}}l_4.$$

Hence $A.\alpha_1\alpha_2\alpha_3$ is the isopole of the plane through $\alpha_1$, $\alpha_2$, $\alpha_3$.

10. Let the points $\alpha_1$, $\alpha_2$, $\alpha_3$ be collinear. Then any point on the intersection of

$$\left.\begin{array}{l} S.\rho\alpha_1 = 0, \\ S.\rho\alpha_2 = 0 \end{array}\right\} \tag{14}$$

is satisfied by the points on the plane $S.\rho\alpha_3 = 0$, and either of these two, for $\alpha_3 = m'\alpha_1 + n'\alpha_2$. Hence if the isopoles of three planes are collinear, they are also collinear.

11. The point of intersection of the three planes

$$S\rho\alpha_1 = 0,\ S\rho\alpha_2 = 0,\ S\rho l_1 = 0$$

is

$$\rho = A.l_1\alpha_1\alpha_2. \tag{15}$$

That of intersection of

$$S\rho\alpha_1 = 0,\ S\rho\alpha_2 = 0,\ S\rho l_2 = 0,$$

is

$$\rho = A.l_2\alpha_1\alpha_2. \tag{16}$$

Hence the line of intersection of these two planes is

$$A.rAl_1\alpha_1\alpha_2\,A.l_2\alpha_1\alpha_2 = 0, \tag{17}$$

and for $l_1$, $l_2$ may be written any two points in space. Any point on this line is

$$\rho = A.\alpha_1\alpha_2\alpha_x, \tag{18}$$

where $\alpha_x$ is any point in space.

12. The line of intersection of the isopolars of points on this line is

$$A.\rho\,(A.lAl_1\alpha_1\alpha_2\,Al_2\alpha_1\alpha_2)(A.l_2Al_1\alpha_x\alpha_2Al_2\alpha_1\alpha_2) = 0 = A.\rho\alpha_1\alpha_2.$$

Any point on this latter line is then

$$\rho = A.\alpha_x\,Al_1\alpha_1\alpha_2\,Al_2\alpha_1\alpha_2, \tag{19}$$

where $\alpha_x$ is any point in space.

Hence if there is given any line,

$$A.r\alpha_1\alpha_2 = 0,$$

there is a second line

$$A\,.pA\,.\,l_r a_1 a_2\,A l_s a_1 a_2 = 0\,, \tag{20}$$

which may be called its isopolar, each possessing the property that all planes through it have their isopoles on the other.

If two lines are isopolars, as

$$A\,.\,r a_1 a_2 = 0\,,$$
$$A\,.\,r a_3 a_4 = 0\,,$$

then for the first any point

$$r = x' a_1 + y' a_2\,.$$

Hence

$$\left.\begin{aligned} x' S a_1 a_3 + y' S a_2 a_3 = 0\,, \\ x' S a_1 a_4 + y' S a_2 a_4 = 0\,, \end{aligned}\right\} \tag{21}$$

$$\begin{aligned} \therefore\ (S a_1 a_3\, S a_2 a_4 - S a_1 a_4\, S a_2 a_3) &= 0 \\ &= S.\,Va_1\,Va_3\,SVa_2\,Va_4 - S.\,Va_1\,Va_4 SVa_2\,Va_3 = S\,.\,A a_1 a_2\,A a_3 a_4\,. \end{aligned} \tag{22}$$

This is the condition of line-isopolarity.

The vector $A\,.\,a_1 a_2$ I shall call the *Pluecker* of the line through $a_1$, $a_2$. For, the six coefficients in the expression

$$\begin{aligned} A\,.\,a_1 a_2 = A l_1 l_2 \,|a_1' a_2''| + A\,.\,l_1 l_3\,|a_1' a_2'''| + A\,.\,l_1 l_4\,|a_1' a_2^{\text{IV}}| \\ + A\,.\,l_2 l_3\,|a_1'' a_2'''| + A\,.\,l_2 l_4\,|a_1'' a_2^{\text{IV}}| + A\,.\,l_3 l_4\,|a_1''' a_2^{\text{IV}}| \end{aligned}$$

are Pluecker's coordinates of a line. They are connected by the relation

$$|a_1' a_2''|\,.\,|a_1''' a_2^{\text{IV}}| - |a_1' a_2'''|\,.\,|a_1'' a_2^{\text{IV}}| + |a_1' a_2^{\text{IV}}|\,.\,|a_1'' a_2'''| = 0\,.$$

Evidently

$$\begin{aligned} S\,.\,l_1 A a_1 a_2 &= \tfrac{1}{2}\,\theta^{\frac{1}{2}}\,(\quad P_{23} + P_{24} + P_{34})\,, \\ S\,.\,l_2 A a_1 a_2 &= \tfrac{1}{2}\,\theta^{\frac{1}{2}}\,(- P_{13} - P_{14} + P_{34})\,, \\ S\,.\,l_3 A a_1 a_2 &= \tfrac{1}{2}\,\theta^{\frac{1}{2}}\,(\quad P_{12} - P_{14} - P_{24})\,, \\ S\,.\,l_4 A a_1 a_2 &= \tfrac{1}{2}\,\theta^{\frac{1}{2}}\,(\quad P_{12} + P_{13} + P_{23})\,, \\ P_{12} P_{34} - P_{13} P_{24} &+ P_{14} P_{23} = 0\,. \end{aligned}$$

These five equations and the fact that only the ratios of the *p*'s is needed, give sufficient data for their determination.

Evidently $A\,.\,a_1 a_2$ is on the line

$$A\,.\,l_r a_1 a_2\,,\quad A\,.\,l_s a_1 a_2\,.$$

The theory of complexes and congruences is developable from these equations.

13. The intersection of

$$Spa_1 = 0\,,$$
$$Spa_2 = 0\,,$$
$$Spa_3 = 0$$

is

$$p = A\,.\,a_1a_2a_3. \tag{23}$$

Hence the point of intersection of three planes is the isopole of the plane of their isopoles.

14. The plane

$$S\,.\,gp = Sp = 0$$

is the plane at infinity, since it is the polar as well as the isopolar of the centroid of the tetrahedron.

Two planes are parallel when they intersect only on the plane at infinity. Hence if they are $S\,.\,a_1p = 0$ and $Sa_2p = 0$,

$$S\,.\,g\,(x'A\,.\,a_ya_1a_2 + y'A\,.\,a_za_1a_2) = 0\,, \tag{24}$$

where $x'$ and $y'$ have any values and $a_y$, $a_z$ are any points. This is satisfied only by

$$\left.\begin{aligned} S\,.\,A\,.\,a_ya_1a_2 = 0\,, \\ S\,.\,A\,.\,a_za_1a_2 = 0\,, \end{aligned}\right\} \tag{25}$$

and since $a_y$ and $a_z$ are arbitrary, it follows that

$$A\,.\,a_1a_2 = 0\,. \tag{26}$$

This is the condition of parallelity of the planes, viz. that the Pluecker of the line joining their isopoles be zero. It also gives

$$a_1 = x'a_2 + y'g\,. \tag{27}$$

Hence if two planes are parallel their isopoles are collinear with the centroid.

15. Two lines $A\,.\,pa_1a_2 = 0$ and $A\,.\,pa_3a_4 = 0$

intersect if

$$S\,.\,a_1Aa_2a_3a_4 = 0\,. \tag{28}$$

If $p$ is the point of intersection,

$$p = xa_1 + ya_2\,,$$
$$\therefore\ xA\,.\,a_1a_3a_4 + yAa_2a_3a_4 = 0\,,$$
$$\therefore\ x : y = Sa_2Aa_3a_4 : \ - Sa_1\,Aa_3a_4\,, \tag{29}$$
$$\left.\begin{aligned} \therefore\quad p &= a_1Sa_2\,Aa_3a_4 - a_2Sa_1\,Aa_3a_4 \\ &= a_3Sa_4\,Aa_1a_2 - a_4Sa_3\,Aa_1a_2\,. \end{aligned}\right\} \tag{30}$$

They are parallel if $Sp = 0$, i. e. if

$$\left.\begin{aligned} & Sa_1 . S . a_2 Aa_3 a_4 - Sa_2 . S . a_1 Aa_3 a_4 = 0 , \\ \text{or} \quad & Sa_3 . S . a_4 Aa_1 a_2 - Sa_4 . S . a_3 Aa_1 a_2 = 0 . \end{aligned}\right\} \tag{31}$$

16. The perpendiculars let fall upon the plane $S . ap = 0$ from the vertices $l_1$, $l_2$, $l_3$, $l_4$ are as

$$S . al_1 : Sal_2 : Sal_3 : Sal_4 . \tag{32}$$

For, these perpendiculars are parallel, hence we have a series of similar triangles which enable us to write out the relations at once.

The perpendicular from any point $q$ is then to that from $l_1$ as

$$S . qa : Sl_1 a , \tag{33}$$

for the perpendicular from $q$ is

$$\begin{gathered} Q_a = x' A_a + y' B_a + z' C_a + w' D_a , \\ \therefore \ \frac{Q_a}{A_a} = \frac{Sqa}{Sl_1 a} . \end{gathered} \tag{34}$$

Hence any two perpendiculars as from $P$ and $Q$ are as

$$Spa : Sqa .$$

17. The volume of $(a_1, a_2, a_3, a_4)$ is

$$\pm \theta \Delta . Sa_1 Aa_2 a_3 a_4 . \tag{35}$$

For the volume

$$\begin{aligned} (a_1 - l_2 l_3 l_4) : (l_1 - l_2 l_3 l_4) &= S . a_1 Al_2 l_3 l_4 : S . l_1 Al_2 l_3 l_4 , \\ (a_2 - l_3 l_4 a_1) : (l_2 - l_3 l_4 a_1) &= S . a_2 Al_3 l_4 a_1 : S . l_2 Al_3 l_4 a_1 , \\ (a_3 - l_4 a_1 a_2) : (l_3 - l_4 a_1 a_2) &= S . a_3 Al_4 a_1 a_2 : S . l_3 Al_4 a_1 a_2 , \\ (a_4 - a_1 a_2 a_3) : (l_4 - a_1 a_2 a_3) &= S . a_4 Aa_1 a_2 a_3 : S . l_4 Aa_1 a_2 a_3 . \end{aligned}$$

Multiplying all together and representing the volume of the reference tetrahedron by $\Delta$,

$$(a_1 - a_2 a_3 a_4) = \pm \theta \Delta . Sa_1 Aa_2 a_3 a_4 .$$

18. A dual interpretation is easily introduced at this point, for if in the present notation

$$S . ap = 0$$

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[NOTE.—$a_1$, $a_2$, $a_3$, $a_4$ must be unit points.]

denote a plane, we may assume an interpretation as follows: Let $a$ represent the plane whose perpendicular distances from $ABCD$ are as the elements of $a$, $a'$, $a''$, $a'''$, $a^{\text{IV}}$, then

$$S.ap = 0$$

will represent a point which is the isopole of this plane. The plane of the three points

$$S.pa_1 = 0, \quad S.pa_2 = 0, \quad S.pa_3 = 0$$

will then be $A.a_1a_2a_3$, etc.

19. If $a'\, b'\, c'\, d'$ represent the faces of the tetrahedron, we may define any conjugate of $P$ as

$$p = \Sigma x' l_1,$$

$$P_r = \Sigma \frac{r'}{x'} l_1 .$$

Many theorems of plane geometry are thus easily extended to solid geometry. I can only hint at them within the limits of this paper.

## THE QUADRIC.

20. The general equation of the quadric is

$$S.p\Phi p = 0,$$

where $\Phi$ is a self-transverse linear quaternion operator. The line $p = a + x'b$ meets this surface at points corresponding to the two roots of the equation

$$x'^2 Sb\Phi b + 2x' Sa\Phi b + Sa\Phi a = 0. \qquad (36)$$

The product of the roots is $\frac{Sa\Phi a}{Sb\Phi b}$; their sum, $-\frac{2Sa\Phi b}{Sb\Phi b}$. The roots are equal if

$$S.a\Phi a\, Sb\Phi b - S^2.a\Phi b = 0. \qquad (37)$$

Hence if $b$ is a variable point satisfying this equation, we have the tangent cone with $a$ as vertex.

This cone touches the quadric at points satisfying the equation (37) and $Sb\Phi b = 0$, i. e. on the plane

$$S.b\Phi a = 0. \qquad (38)$$

This is the polar plane of $a$. If $a$ is on the surface, it is the tangent plane.

The pole of $S.ap = 0$ is evidently therefore

$$\Phi^{-1}a. \tag{39}$$

Hence the center, or pole of the plane at infinity is

$$\Phi^{-1}g. \tag{40}$$

This is at infinity, and the surface is a paraboloid if

$$S.g\Phi^{-1}g = 0. \tag{41}$$

21. If one root of $\Phi$ is zero, then $m = 0$; and since

$$\Phi A.\Phi l_1 \Phi l_2 \Phi l_3 = 0,$$

in this case, $A.\Phi l_1 \Phi l_2 \Phi l_3$ must be a point on the surface. And since

$$S.p\Phi A.\Phi l_1 \Phi l_2 \Phi l_3 = 0,$$

for all values of $p$, i. e.

$$S.\Phi pA.\Phi l_1 \Phi l_2 \Phi l_3 = 0,$$

$A.\Phi l_1 \Phi l_2 \Phi l_3$ must lie on all polar planes, hence on all tangent planes. If $p$ is any other point on the surface,

$$A.\phi l_1 \phi l_2 \phi l_3 + x'p$$

evidently satisfies the equation of the surface. It must therefore be a cone whose vertex is

$$A.\Phi l_1 \Phi l_2 \Phi l_3.$$

In case two roots are zero, we have likewise $\Phi^2 \mathrm{X} l_1 = 0$, hence $\Phi \mathrm{X} l_1$ is a point on the surface. All tangent planes and all polar planes then go through two points, and the surface is therefore two planes.

In case three roots are zero, we find likewise that $\Phi^2 \Psi l_1$ is a point of the surface, which must then be one plane.

22. If the polar plane of $a$ pass through $b$, $Sa\Phi b = 0$, and this is the sole condition that the polar plane of $b$ pass through $a$.

23. The asymptotic cone is clearly

$$S.g\Phi^{-1}g\, Sp\Phi p - S^2 gp = 0, \tag{42}$$

or since $g$ is a scalar,

$$S.1\Phi^{-1}1.Sp\Phi p - S^2 p = 0. \tag{43}$$

24. The plane $Sap = 0$ touches the surface if it contain its own pole, hence the condition is

$$S\,.\,a\Phi^{-1}a = 0. \tag{44}$$

25. The polar line of $$A\,.\,pab = 0$$ is easily found. For the polar plane of $a$ is $Sp\Phi a = 0$, and of $b$ is $Sp\Phi b = 0$. Hence the polar line is

$$A\,.\,rAa_x\Phi a\,\Phi b\,A\,.\,a_y\,\Phi a\Phi b = 0. \tag{45}$$

26. Referred to the invariant points of $\Phi$, $p_1, p_2, p_3, p_4$ we see at once that since

$$\Phi p_1 = p_1, \quad Sp_1p_2 = 0, \text{ etc.}$$

the invariant tetrahedron is self-polar with respect to the surface.

The usual theory is easily worked out from these first principles.

ILLINOIS COLLEGE, *August* 1, 1895.